Lorentz Contraction of Bound States in 1+1 Dimensional Gauge Theory

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Abstract

We consider the Lorentz contraction of a fermion-antifermion bound state in 1+1 dimensional QED. In 1+1 dimensions the absence of physical, propagating photons allows us to explicitly solve the weak coupling limit $\alpha \ll m^2$ of the Bethe-Salpeter bound state equation in any Lorentz frame. In a time-ordered formalism it is seen that all pair production is suppressed in this limit. The wave function is shown to contract while the mass spectrum is invariant under boosts.

1   Introduction

The aim of this paper is to study the boost properties of bound state equal time wave functions in gauge field theories. We will show explicitly how the fermion-antifermion wave functions Lorentz contract in the case of 1+1 dimensional QED in the weak coupling limit $e^2 \ll m^2$. The solution is easier in 1+1 dimensions than in 3+1 dimensions where contributions from propagating gauge fields must be taken into account. To our knowledge, the exact Lorentz contraction in gauge field theories has not been demonstrated explicitly before even in the 1+1 dimensional case. In [1] a Lorentz contracting wave function of a two body bound state in (3+1 dimensional) QED is represented as an approximation valid for small boosts. The frame dependence of bound state wave functions has also been studied previously in various models, see for example [2, 3]. In [4] Lorentz contraction is obtained for a fermion pair interacting via a δ-potential. The frame dependence of meson wave functions in the Gross-Neveu model is studied in [5]. There has also been other interesting work on the Lorentz covariance of two body equations [6, 7].

In the standard instant form quantization of the theory, the wave functions are evaluated with all constituents having the same time x^0. Another possibility is to use the light-front quantization: the theory is quantized at equal light-front time $x^+ = (x^0 + x^1)/\sqrt{2}$ [8]. In that case it is natural to study bound state wave functions evaluated at equal x^+ instead of x^0.

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Special relativity leads to Lorentz contraction of classical objects. According to the Lorentz transformation rules, a moving object contracts in the direction of motion: the spatial distance between two points of the object is proportional to $1/\gamma$ when measured at equal x^0 in the observer's frame. Similarly, the wave functions evaluated at equal x^0 of the constituents are expected to Lorentz contract under boosts. The boost generators K^i do not commute with the Hamiltonian for translations in x^0 and thus the boosts are dynamical. Consequently, the equal x^0 bound state wave function of the center-of-mass frame cannot be kinematically boosted to a general frame. This is why its contraction is non-trivial.

The non-trivial transformation under boosts of wave functions defined at equal x^0 is consistent with the Lorentz covariant formulation of bound state equations (Bethe-Salpeter formalism [9]). *E.g.*, a covariant two-fermion wave function for a bound state $|\mathbf{P}\rangle$ with 3-momentum $\mathbf{P}$ is defined by

$$\chi_{\mathbf{P}}(X,x)_{\alpha\beta} = \langle\Omega|\mathrm{T}\left\{\bar{\psi}_\beta(X-x/2)\psi_\alpha(X+x/2)\right\}|\mathbf{P}\rangle \ . \tag{1}$$

where x is the relative coordinate. The Lorentz transformation of this wave function (for a scalar state) is given by [1]

$$\chi_{\mathbf{P'}}(X',x') = S(\Lambda)\,\chi_{\mathbf{P}}(X,x)\,S^{-1}(\Lambda) \tag{2}$$

where $x' = \Lambda x$, $P' = \Lambda P$ and $S(\Lambda)$ is the usual spin transformation matrix. For boosts we cannot have both $x^0 = 0$ and $x'^0 = 0$. Thus (2) does not specify a transformation between equal x^0 wave functions. However, in [1] it is suggested that the x^0 dependence of the wave function can be omitted in the weak coupling limit for boosts from the center-of-mass frame to a frame with small momentum $\mathbf{P}$. We are going to discuss the validity of this approximation here.

The dynamics involved in boosting equal x^0 wave functions can be qualitatively seen by considering the hydrogen atom, which (to a good approximation) is non-relativistic in the rest frame. In this frame the typical scale of the kinetic energy of the electron is $\alpha^2 m_e$ and its typical momentum is of order αm_e. Thus, by momentum conservation, the momenta of exchanged physical (transverse) photons are (at least) αm_e. Hence $E_\gamma = |\mathbf{p}_\gamma| = \alpha m_e \gg E_e \approx \alpha^2 m_e/2$ so that Fock states with transverse photons may be neglected at lowest order.

In a boosted frame the situation is different. The electron carries approximately a fraction $m_e/(m_e+m_p) \equiv a$ of the total center-of-mass momentum P. Let the electron momenta parallel and perpendicular to the boost be $aP + p_\parallel$ and $p_\perp$, respectively. For $aP \gg m_e$, the kinematically boosted electron energy is

$$E = \sqrt{(aP+p_\parallel)^2 + p_\perp^2 + m_e^2} \approx aP + p_\parallel + \frac{(m_e)^2 + p_\perp^2}{2aP} \ . \tag{3}$$

The magnitude of the electron energy fluctuations is given by the second term $p_\parallel$, which also is the typical photon momentum scale. Thus in the boosted frame the photon and electron energy fluctuations are of the same order and Fock states with additional photons can be relevant even at lowest order. The wave function at equal x^0 of the hydrogen atom in fact has not (to our knowledge) been calculated in a general frame.

We will here study Lorentz contraction in the particularly simple case of 1+1 dimensional QED at weak coupling, $e^2 \ll m_e^2$. In 1+1 dimensions, the time derivatives of

the photon field vanish from the Lagrangian in the gauge $A^1 = 0$. Thus there are no physical, propagating photons. The photon propagator becomes instantaneous, it only connects fermions having the same x^0. Our study applies as such to QCD_2 which has the same Feynman rules as QED_2 up to colour factors in the $A^{a\,1} = 0$ gauge where the gluon self interactions are absent.

Lorentz covariance is nevertheless non-trivial even in 1+1 dimensions. This is illustrated by the fact that the "relativistic" Schrödinger two body equation with the Hamiltonian

$$H = \sqrt{(p_1^1)^2 + m_1^2} + \sqrt{(p_2^1)^2 + m_2^2} + V(|x_1^1 - x_2^1|) \tag{4}$$

and a linear (Coulomb) potential is known not to be Lorentz covariant [7]: Its energy eigenvalues do not have the correct dependence on the CM momentum, $E \neq \sqrt{M^2 + (P^1)^2}$. This indicates the importance of including Fock states with additional particle pairs. Here we eliminate their contribution by restricting ourselves to the weak coupling limit of QED, $e^2 \ll m_e^2$. In this limit we show that the eigenvalues of (4) indeed satisfy $E = \sqrt{M^2 + (P^1)^2}$.

2   Lorentz Contraction in QED$_2$

We show here that in the weak coupling limit the fermion-antifermion equal x^0 wave function in 1+1 dimensional QED (QED$_2$) Lorentz contracts while the mass spectrum of the equation is invariant in boosts. The QED$_2$ spectrum is affected by the presence of a background electric field — the θ parameter of Ref. [10]. Since the field strength F_{01} is invariant under boosts we may limit ourselves to the simplest case where the background field vanishes in all frames. We also take the fermion masses to be equal for notational simplicity.

We use the $A^1 = 0$ gauge. In this gauge, the photon propagators are infra-red singular. These singularities are handled using the principal value prescription (see, *e.g.*, [11]). The propagator is defined to be

$$D(k^1) \equiv \mathrm{P}\frac{i}{(k^1)^2} \equiv \frac{1}{2}\frac{i}{(k^1 + i\epsilon)^2} + \frac{1}{2}\frac{i}{(k^1 - i\epsilon)^2} \; . \tag{5}$$

Then the Fourier transform of the propagator is the linear Coulomb potential

$$\int \frac{\mathrm{d}k^1}{2\pi} iD(k^1)e^{ik^1 x^1} = -\mathrm{P}\int \frac{\mathrm{d}k^1}{2\pi}\frac{1}{(k^1)^2}e^{ik^1 x^1} = \frac{1}{2}|x^1| \; . \tag{6}$$

Note that (5), (6) define a principal value integral for a double pole.

We consider the weak coupling limit of the Bethe-Salpeter bound state equation [9] for QED$_2$ and an arbitrary Lorentz frame. This straightforward calculation is done more in detail in [12]. The equation is shown diagrammatically in fig. 1. In particular, S is the full two particle propagator including radiative corrections (as in [11]). The interaction kernel K contains all possible (irreducible) interaction graphs. The total two-momentum of the bound state is denoted by P. The wave function Ψ_{P^1} is defined as the coupling of the bound state $|P^1\rangle$ to a fermion pair

$$\Psi_{P^1}(p) = \int \mathrm{d}^2x \, e^{ix \cdot p}\langle\Omega|\mathrm{T}\left\{\bar\psi(0)\psi(x)\right\}|P^1\rangle \tag{7}$$
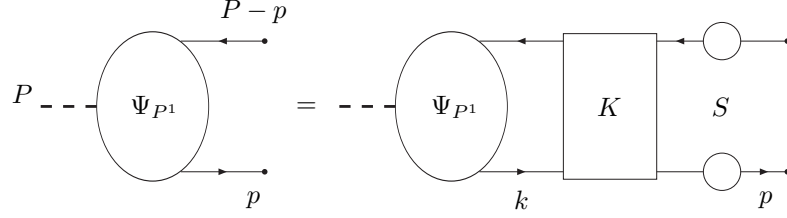
3

Figure 1: The Bethe-Salpeter equation. The blobs represent the wave function Ψ_{P^1}, K is the interaction kernel and S is the two-particle propagator.

where $|\Omega\rangle$ is the vacuum of the theory. The equation reads

$$\Psi_{P^1}(p) = S(P,p) \int \frac{\mathrm{d}^2 k}{(2\pi)^2} K(P,p,k)\Psi_{P^1}(k) \ . \tag{8}$$

Ψ_{P^1}, S and K also have Dirac indices which were hidden above.

Although the equation (fig. 1) is for the coupling of the bound state to two particles, there still is pair production and Fock states with infinitely many particles involved in K and S. In order to make this more explicit we now move to a time-ordered formalism by taking the Fourier transform over p^0. In the gauge $A^1 = 0$ the photon propagator is independent of energy: the interaction is instantaneous, connecting fermions at the same instant of time. This makes the time-ordered formalism particularly simple. Taking the Fourier transform to reach (x^0, p^1) space, the photon and fermion propagators become

$$D(x^0, k^1) \equiv \delta(x^0)\,\mathrm{P}\frac{i}{(k^1)^2} = \delta(x^0)D(k^1)$$

$$S_F(x^0, p^1) \equiv \Theta(x^0)\Lambda^+(p^1)e^{-i\,x^0 E_{p^1}} + \Theta(-x^0)\Lambda^-(-p^1)e^{i\,x^0 E_{p^1}} \tag{9}$$

where the projection operators $\Lambda^{\pm}$ are defined by

$$\Lambda^{\pm}(p^1) \equiv \frac{\pm\gamma^0 E_{p^1} \mp \gamma^1 p^1 + m}{2E_{p^1}} \tag{10}$$

and $E_{p^1} \equiv \sqrt{m^2 + (p^1)^2}$. In addition to the usual momentum space Feynman rules, there are now integrations over the time of each vertex. These integrations give energy denominators.

We wish to find the leading terms of the Bethe-Salpeter equation in the weak coupling limit. To do this we need to find the correct expansion parameter. In 1+1 dimensions, the non-relativistic two particle Hamiltonian for a system interacting via a Coulomb force is, in terms of the relative coordinates and in the center-of-mass frame,

$$H = \frac{1}{2\mu}(p^1)^2 + \frac{e^2}{2}|x^1| \tag{11}$$

where $\mu = m/2$. Equating the (relative) kinetic and potential parts we have $\alpha|x^1| \sim \alpha/p^1 \sim (p^1)^2/m$, hence

$$p^1 \sim \left(\frac{\alpha}{m^2}\right)^{1/3} m \ , \quad V \sim \left(\frac{\alpha}{m^2}\right)^{2/3} m \ . \tag{12}$$

$$-i\Sigma(x^0, p^1) \quad = \quad -i\delta(x^0)\Sigma(p^1) \quad = \quad$$

Figure 2: One-loop radiative correction to the quark propagator. The δ-function comes from the instantaneuos photon propagator which is denoted by a dashed line.

Thus in 1+1 dimensions the natural dimensionless expansion parameter is $p^1/m \sim \left(\alpha/m^2\right)^{1/3}$, and we will calculate to first non-trivial order in this parameter.

In a general Lorentz frame with a total center-of-mass momentum P, we expect contraction effects in addition to (12):

$$p^1 \sim \left(\frac{\alpha}{m^2}\right)^{1/3} m \cdot \frac{\mathcal{E}}{m} \;, \quad V \sim \left(\frac{\alpha}{m^2}\right)^{2/3} m \cdot \frac{m}{\mathcal{E}} \tag{13}$$

where p^1 is now the relative, internal momentum and $\mathcal{E} \equiv \sqrt{(2m)^2 + (P^1)^2}$.

2.1 Relevant Diagrams for the Equation in the Weak Coupling Limit

We will now show that in the Bethe-Salpeter equation we can replace the full propagator S by the free propagator and the kernel K by single photon exchange in the weak coupling limit in all Lorentz frames. First we analyze the propagator S. We check explicitly the one loop correction (fig. 2).

In the center-of-mass frame the result for $\Sigma(p^1)$ is easy to foresee. In the coordinate space, the instantaneous fermion propagator vanishes like $\exp(-m|x^1|)$ and the fermion can only propagate a distance $1/m$, a magnitude smaller than the size of the bound state $x^1 \sim 1/p^1 \sim (m^2/\alpha)^{1/3}1/m$ from (12). Thus the fermion feels the potential of the photon at the distance $1/m$ and the magnitude of the self-energy correction will be similarly a magnitude smaller than the binding energy scale $\Delta E \sim V$ from (12),

$$\Sigma(p^1) \sim V(x^1 \sim 1/m) \sim \alpha/m^2 \cdot m \ll \Delta E \sim \left(\alpha/m^2\right)^{2/3} \cdot m \;. \tag{14}$$

$\Sigma(p^1)$ contributes a shift in the fermion energies and hence a shift in the final binding energy, hence the comparison with ΔE makes sense.

For a general Lorentz frame it is more convenient to do the exact calculation. Using (x^0, p^1) space Feynman rules, we get

$$
\begin{aligned}
-i\Sigma(p^1) \quad &= \quad -2\alpha \int \mathrm{d}k^1 D(p^1 - k^1)\gamma^0 S_F(0, k^1)\gamma^0 \\
&= \quad -i\alpha\,\mathrm{P} \int \mathrm{d}k^1 \frac{\gamma^1 k^1 + m}{(k^1 - p^1)^2 \sqrt{(k^1)^2 + m^2}} \;.
\end{aligned}
\tag{15}
$$

where P denotes the principal value and the instantaneous fermion propagator is

defined by[2]

$$S_F(0, k^1) = \frac{1}{2}\left(S_F(\epsilon, k^1) + S_F(-\epsilon, k^1)\right) = \frac{-\gamma^1 k^1 + m}{2E_{k^1}} \ . \tag{16}$$

The k^1 integral can be done using, *e.g.*, Mathematica:

$$-i\Sigma(p^1) = -2i\alpha \left(-\frac{\gamma^1 p^1 + m}{(p^1)^2 + m^2} + \frac{m(\gamma^1 m - p^1)}{2((p^1)^2 + m^2)^{3/2}} \log \frac{\sqrt{(p^1)^2 + m^2} - p^1}{\sqrt{(p^1)^2 + m^2} + p^1} \right) \ . \tag{17}$$

In the center-of-mass frame p^1 is small w.r.t. the fermion mass, and we can expand in powers of p^1/m:

$$-i\Sigma(p^1) = -2i\alpha \left(-\frac{1}{m} - 2\gamma^1 \frac{p^1}{m^2} \right) \left(1 + \mathcal{O}\left(\frac{(p^1)^2}{m^2} \right) \right) \ . \tag{18}$$

Wee see that the leading contribution is $\propto \alpha/m$, as expected. The scaling result for any P^1 is rather complicated, but the order of the expression is smaller than or equal to $\alpha/\mathcal{E}$ (in a general frame $p^1 \sim P^1$). The result should again be compared with the scale of the potential or the binding energy ΔE (see (13)):

$$\Delta E \sim \left(\frac{\alpha}{m^2} \right)^{2/3} m \cdot \frac{m}{\mathcal{E}} = \left(\frac{m^2}{\alpha} \right)^{1/3} \frac{\alpha}{\mathcal{E}} \tag{19}$$

which is larger than $\Sigma(p^1)$ by a factor of $(m^2/\alpha)^{1/3}$ for all P^1. Hence $\Sigma(p^1)$ is negligible.

Next we will analyze the structure of K in more detail. Iterating the bound state equation (fig. 1) gives the solution as a sum of infinite ladders of diagrams. When time-ordered, the ladders contain blocks of which a representative set is shown in fig. 3. Blocks in fig. 3 a) and c) consist of single photon exchange diagrams. The diagram in fig. 3 b) is irreducible and comes from a "crossed" exhange diagram in K with two photons. The diagram in fig. 3 d) is a radiative correction to the fermion-photon vertex. The "Z graphs" b), c) and the diagram d) include pair production and are suppressed. The suppression is caused by the $1/\Delta E$ terms arising from the ΔX^0 integration: we have, *e.g.*,

$$\begin{aligned} \text{a)} \quad &\propto \quad \frac{1}{E - E_{P^1 - p^1 + k^1} - E_{p^1 - k^1}} \\ \text{b)} \quad &\propto \quad \frac{1}{E - E_{P^1 - p^1} - E_{P^1 - q^1 - k^1} - E_{P^1 - q^1} - E_{p^1 - k^1}} \end{aligned} \tag{20}$$

where E is the bound state energy. The factors multiplying (20) are of the same order for both diagrams a) and b). For diagram a) we have a cancellation

$$E - E_{P^1 - p^1 + k^1} - E_{p^1 - k^1} \sim \Delta E \sim \left(\frac{\alpha}{m^2} \right)^{2/3} m \cdot \frac{m}{\mathcal{E}} \tag{21}$$

which does not occur in the case of diagram b), for which the energy difference is $\mathcal{O}(\mathcal{E})$. Thus diagram b) is suppressed. A similar calculation shows the suppression for diagrams c), d) and also for more complicated (irreducible) diagrams with more than two photons, not included in fig. 3. The result is that we can neglect all terms but the single photon exchange in K in the weak coupling limit in any Lorentz frame.

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[2]In fact, the result for $\Sigma(p^1)$ is not sensitive to how the instantaneous propagator is defined: the discontinous γ^0 term vanishes in the k^1 integration.
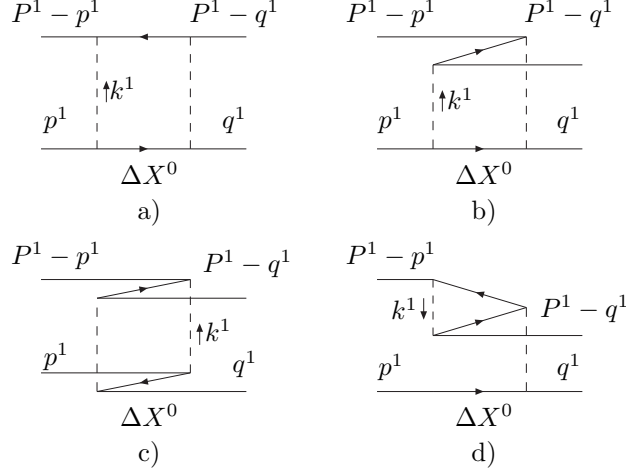
Figure 3: Examples of ladder diagrams which are present in the full bound state equation. $\Delta X^0 > 0$ and time flows from left to right.

2.2 The Wave Equation in the Weak Coupling Limit

According to the previous section we can use the equation of fig. 1 with free propagators and only single photon exchange in any Lorentz frame. The result is shown diagrammatically in time ordered form in fig. 4. The analytic expression is

$$\varphi_{P^1}(p^1) = \frac{2\alpha}{E - E_{p^1} - E_{P^1 - p^1} + i\epsilon} \Lambda^+(p^1)\gamma^0 \; \text{P} \int \frac{\mathrm{d}k^1}{(k^1)^2} \varphi_{P^1}(p^1 - k^1)\gamma^0 \Lambda^-(P^1 - p^1)$$

$$+ \frac{2\alpha}{E + E_{p^1} + E_{P^1 - p^1} + i\epsilon} \Lambda^-(p^1)\gamma^0 \; \text{P} \int \frac{\mathrm{d}k^1}{(k^1)^2} \varphi_{P^1}(p^1 - k^1)\gamma^0 \Lambda^+(P^1 - p^1) \qquad (22)$$

Here E is the bound state energy and φ is the equal time wave function

$$\varphi_{P^1}(p^1) \equiv \int \frac{\mathrm{d}p^0}{2\pi} \Psi_{P^1}(p) \qquad (23)$$

which is a 2x2 matrix in Dirac space in the notation of eq. (22).

When iterated, the last term on the right hand side of (22) will generate "Z graphs" with pair production (fig. 3 c)). As shown in the previous section this contribution is suppressed in the weak coupling limit, and the last term can be neglected. This can also be seen directly from the $1/\Delta E$ terms in equation (22).

Keeping only the leading term, the equation reads

$$(E - E_{p^1} - E_{P^1 - p^1})\varphi_{P^1}(p^1) = 2\alpha\Lambda^+(p^1)\gamma^0 \left[\text{P} \int \frac{\mathrm{d}k^1}{(k^1)^2} \varphi_{P^1}(p^1 - k^1) \right] \gamma^0 \Lambda^-(P^1 - p^1) \qquad (24)$$

From (24) we see that the wave function φ_{P^1} is proportional to the projection matrices $\Lambda^\pm$ which can be used to further simplify the equation. When multiplying (24) with $\Lambda^+(P^1 - p^1)\gamma^0$ from the right and $\gamma^0 \Lambda^-(p^1)$ from the left we get, using $\Lambda^+ \Lambda^- = 0 = \Lambda^- \Lambda^+$,

$$
\begin{aligned}
E_p \varphi_{P^1}(p^1) &= \left[\gamma_5 p^1 + \gamma^0 m \right] \varphi_{P^1}(p^1) \\
E_{P-p} \varphi_{P^1}(p^1) &= \varphi_{P^1}(p^1) \left[-\gamma_5(P^1 - p^1) - \gamma^0 m \right]
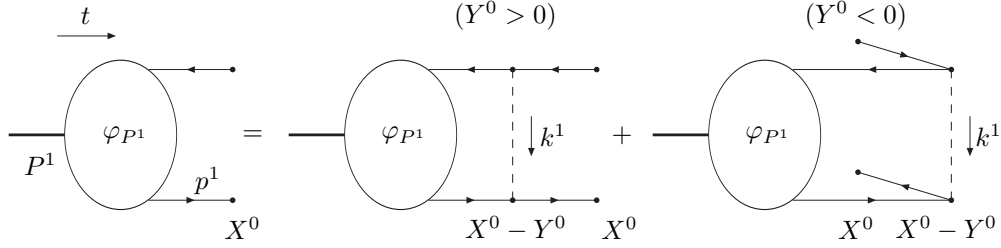\end{aligned}
\qquad (25)
$$

7

Figure 4: The time-ordered bound state equation. The blobs denote the equal x^0 wave function φ_{P^1}.

where $\gamma_5 \equiv \gamma^0 \gamma^1$. These equations may be used to eliminate the terms proportional to P^1, p^1 and m in the projection operators of (24), giving on the right hand side

$$- 2\alpha \, \mathrm{P} \int \frac{\mathrm{d}k^1}{(k^1)^2} \frac{\left(E_{p^1} + E_{p^1-k^1}\right) + \gamma_5 k^1}{2 E_{p^1}} \varphi_{P^1}(p^1 - k^1) \frac{\left(E_{P^1-p^1} + E_{P^1-p^1+k^1}\right) + \gamma_5 k^1}{2 E_{P^1-p^1}} .$$

(26)

Because of the momentum conservation the photon momentum k^1 is of the order of the internal momenta: $k^1 \sim (\alpha/m^2)^{1/3}\mathcal{E} \ll p^1,\ P^1 - p^1$. Hence we can neglect k^1 in the numerator of eq. (26), reducing the projection matrices to unity. (24) becomes

$$(E - E_{p^1} - E_{P^1-p^1})\varphi_{P^1}(p^1) = -2\alpha \, \mathrm{P} \int \frac{\mathrm{d}k^1}{(k^1)^2} \varphi_{P^1}(p^1 - k^1) .$$

(27)

This is the relativistic Schrödinger equation, which is not Lorentz covariant for general α/m^2 [7]. However, further developing also the energies on left hand side of (27) to first order in $\left(\alpha/m^2\right)^{1/3}$ we have

$$E_{P^1-p^1} + E_{p^1} = \mathcal{E} + \frac{(\tilde{p}^1)^2}{2\mu} \frac{(2m)^3}{\mathcal{E}^3} + \mathcal{O}\left((\tilde{p}^1)^3\right) .$$

(28)

where $\mu = m/2$, $\mathcal{E} = \sqrt{(2m)^2 + (P^1)^2}$ and $\tilde{p}^1$ is the relative momentum $\tilde{p}^1 \equiv p^1 - P^1/2$. Fourier transforming into coordinate space we get

$$(E - \mathcal{E})\varphi_{P^1}(x^1) = \left[-\frac{1}{2\mu} \frac{(2m)^3}{\mathcal{E}^3} \left(\frac{\partial}{\partial x^1}\right)^2 + 2\pi\alpha|x^1| \right] \varphi_{P^1}(x^1) .$$

(29)

Up to non-leading terms (29) can be written

$$\Delta M \varphi_{P^1}(x^1) = \left[-\frac{1}{2\mu} \frac{M^2}{E^2} \left(\frac{\partial}{\partial x^1}\right)^2 + 2\pi\alpha \frac{E}{M}|x^1| \right] \varphi_{P^1}(x^1)$$

(30)

where M is the bound state mass and $\Delta M = M - 2m$. We have thus showed that (30) is the weak coupling limit of the Bethe-Salpeter equation (fig. 1) in an arbitrary Lorentz frame. Remarkably, we see that $\varphi_P(x^1/\gamma)$ where $\gamma \equiv E/M$ satisfies an equation which is independent of P, and thus the wave function Lorentz contracts while the spectrum stays invariant. Note that this holds only for a linear potential. Remember that φ_{P^1} is a 2x2 matrix in Dirac space although the Dirac structure does not appear in (30). The Dirac structure of φ_{P^1} is obtained from (24) (see also section 2.4 below).
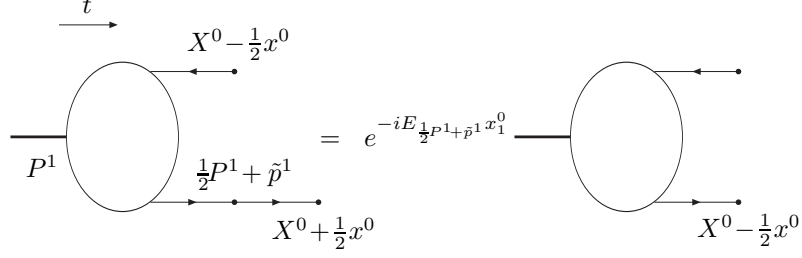
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$$t \longrightarrow$$

$$X^0 - \tfrac{1}{2}x^0$$

$$P^1 \qquad \tfrac{1}{2}P^1 + \tilde{p}^1 \qquad = \quad e^{-iE_{\frac{1}{2}P^1+\tilde{p}^1}\,x_1^0}$$

$$X^0 + \tfrac{1}{2}x^0 \qquad\qquad\qquad X^0 - \tfrac{1}{2}x^0$$

Figure 5: The time dependence of the wave function when one of the fermions is shifted in time $(x^0 > 0)$, neglecting the backward moving fermions. The exponential factor arises from the time development of the fermion propagator. The overall time is X^0 on the left and $X^0 - \tfrac{1}{2}x^0$ on the right.

2.3 Connection to the Covariant Formalism

In the weak coupling limit the motion of the fermion and the antifermion inside the bound state is slow: $v \sim (\alpha/m^2)^{1/3}$. Thus it is not essential that the positions of the constituents are measured at exactly the same x^0 in the center-of-mass frame. For example, wave functions evaluated at equal x^0 and and at equal x^+ are expected to coincide. Another consequence, which we prove below, is that the relative time dependence of the covariant wave function (1) $\chi_{\mathbf{P}=0}$ can be neglected at leading order in $(\alpha/m^2)^{1/3}$. This can be used to study more carefully how the transformation rule (2) can be applied to obtain the Lorentz contraction in the weak coupling limit, *i.e.*, study the validity of the approximation suggested in [1].

We again restrict ourselves to the 1+1 dimensional case with equal fermion masses. The interaction is instantaneous and backward moving fermions are absent which makes it possible to find the relative time dependence of the wave function (1). The time dependence has the simplest form in (x^0, p^1) space using the relative coordinates

$$
\begin{aligned}
\Phi_{P^1}(X^0, x^0, \tilde{p}^1) &\equiv \int dx^1\, e^{-i\tilde{p}^1 x^1} \left. \langle \Omega | T \left\{ \bar{\psi}(X - x/2)\psi(X + x/2) \right\} |P^1\rangle \right|_{X^1=0} \\
&= \int dx^1\, e^{-i\tilde{p}^1 x^1} \left. \chi_{P^1}(X, x) \right|_{X^1=0} .
\end{aligned}
\tag{31}
$$

where translation invariance allows us to put $X^1 = 0$.

The overall time dependence of Φ is given by

$$
\Phi_{P^1}(X^0, x^0, \tilde{p}^1) = e^{-iEX^0} \Phi_{P^1}(0, x^0, \tilde{p}^1)
\tag{32}
$$

where $E = \sqrt{M^2 + (P^1)^2}$, M being the bound state mass. Since the interaction is instantaneous one of the fermions may be shifted in time (see fig. 5). A similar result holds for shifting the antifermion. In terms of Φ_{P^1} we have

$$
\Phi_{P^1}(0, x^0, \tilde{p}^1) = \begin{cases} e^{-iE_{\frac{1}{2}P^1+\tilde{p}^1}\,x^0}\, \Phi_{P^1}(-\tfrac{1}{2}x^0, 0, \tilde{p}^1) & \text{when } x^0 > 0 \\ e^{+iE_{\frac{1}{2}P^1-\tilde{p}^1}\,x^0}\, \Phi_{P^1}(+\tfrac{1}{2}x^0, 0, \tilde{p}^1) & \text{when } x^0 < 0 \end{cases}
\tag{33}
$$

Combining this with (32) gives the the relative time dependence

$$
\Phi_{P^1}(0, x^0, \tilde{p}^1) = \begin{cases} e^{-i(E_{\frac{1}{2}P^1+\tilde{p}^1} - \frac{1}{2}E)x^0}\, \Phi_{P^1}(0, 0, \tilde{p}^1) & \text{when } x^0 > 0 \\ e^{-i(\frac{1}{2}E - E_{\frac{1}{2}P^1-\tilde{p}^1})x^0}\, \Phi_{P^1}(0, 0, \tilde{p}^1) & \text{when } x^0 < 0 \end{cases}
\tag{34}
$$

9

Let us now apply the 1+1 dimensional version of the transformation (2) to a boost from the center-of-mass frame to a frame with momentum P^1 putting $X = 0$ and $x'^0 = 0$:

$$\chi_{P^1}(0, x')\Big|_{x'^0=0} = S(\Lambda)\, \chi_0(0, x)\, S^{-1}(\Lambda) \tag{35}$$

Then we have $x^0 = -\sinh\zeta\, x'^1$ with ζ denoting the rapidity of the boost. Using the estimates (13) we have

$$x^0 \sim \sinh\zeta\, x'^1 \sim P^1/(m\tilde{p}'^1) \sim \left(\alpha/m^2\right)^{-1/3} 1/m \cdot P^1/E \ . \tag{36}$$

For the center-of-mass wave function the energy differences appearing in (34) are small

$$\frac{E}{2} - E_{\pm\tilde{p}^1} \sim \left(\alpha/m^2\right)^{2/3} m \ . \tag{37}$$

Thus the exponents in (34) are $\sim \left(\alpha/m^2\right)^{1/3} P^1/E$ for the center-of-mass wave function Φ_0 and give a next to leading correction. This implies that we can drop the relative time dependence of χ_0 in (35). In terms of the equal x^0 wave function

$$\varphi_{P^1}(x^1) = \chi_{P^1}(0, x)\Big|_{x^0=0} \tag{38}$$

we then have

$$\varphi_{P^1}(x^1) = S(\Lambda)\, \varphi_0(\gamma x^1)\, S^{-1}(\Lambda) \tag{39}$$

which is in agreement with the results of the previous section (see also (41) below).

We conclude that the relative time dependence of χ_0 in (35) can be dropped for all boosts Λ at the leading order in $\left(\alpha/m^2\right)^{1/3}$. If next to leading order corrections are included in the wave functions as in [1], (39) is applicable as such for small P^1/m. For large boosts at next to leading order or for higher order corrections the classical contraction result (39) fails. An analogous result was obtained in the model calculation of Ref. [3], where it was seen that the Lorentz contraction differs from the classical one for $\alpha \gtrsim m^2$.

2.4 QED$_2$ in the Infinite Momentum Frame

It is generally believed (see, e.g., [13]) that the equal x^0 description of dynamical processes coincides, in the infinite momentum frame, with physics on the light-front $(x^+ = 0)$. We may verify this property of our equal x^0 bound state wave functions in frames with $P^1 \gg 2m$.

The result (24) implied that φ_{P^1} is proportional to the projection matrices $\Lambda^\pm$ of (10),

$$\varphi_{P^1}(\tilde{p}^1) = \Lambda^+(P^1/2 + \tilde{p}^1)\cdots\Lambda^-(P^1/2 - \tilde{p}^1) \ . \tag{40}$$

in terms of the relative momentum $\tilde{p}^1 = p^1 - P^1/2$. Taking the contraction result into account, we then have at leading order in $\left(\alpha/m^2\right)^{1/3}$

$$\varphi_{P^1}(\tilde{p}^1)_{\alpha\beta} = u_\alpha(P^1/2)\bar{v}_\beta(P^1/2)\phi(\tilde{p}^1/\gamma) \tag{41}$$

where ϕ is a scalar function which is independent of the frame, and u, v are the Dirac spinors:

$$u(p^1) = \frac{2E_{p^1}}{\sqrt{E_{p^1} + m}} \Lambda^+(p^1) \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$v(p^1) = \frac{2E_{p^1}}{\sqrt{E_{p^1} + m}} \Lambda^-(p^1) \begin{pmatrix} 0 \\ 1 \end{pmatrix} . \tag{42}$$

In a frame where $P^1 \gg 2m$ we get

$$u(P^1/2) = \sqrt{\frac{P^1}{2}} \left[\begin{pmatrix} 1 + \frac{m}{P^1} \\ 1 - \frac{m}{P^1} \end{pmatrix} + \mathcal{O}\left(\frac{m^2}{(P^1)^2} \right) \right]$$

$$v(P^1/2) = \sqrt{\frac{P^1}{2}} \left[\begin{pmatrix} 1 - \frac{m}{P^1} \\ 1 + \frac{m}{P^1} \end{pmatrix} + \mathcal{O}\left(\frac{m^2}{(P^1)^2} \right) \right] \tag{43}$$

that is, the spinors in (41) approach the form of the light front spinors [13] ($P^1 \approx P^+/\sqrt{2}$ for $P^1 \gg 2m$).

The scalar function $\phi(\tilde{p}^1/\gamma)$ satisfies equation (27). Taking the limit $P^1 \to \infty$ and writing $E = \sqrt{M^2 + (P^1)^2}$ we have

$$E - E_{p^1} - E_{P^1 - p^1} = \frac{M^2}{2P^1} - \frac{m^2}{2p^1} - \frac{m^2}{2P^1 - 2p^1} + \mathcal{O}\left(\frac{m^4}{(P^1)^3} \right) . \tag{44}$$

if $0 < p^1 < P^1$. Outside this region, we get an $\mathcal{O}\left(P^1\right)$ term in (44), and inserting this into (27) implies that the wave function must vanish. Equation (27) thus reduces to the 't Hooft equation [11],

$$M^2\phi(x) = m^2 \left(\frac{1}{x} + \frac{1}{1-x} \right) \phi(x) - 4\alpha\, \mathrm{P} \int_0^1 \frac{\mathrm{d}y}{(x-y)^2} \phi(y) . \tag{45}$$

where $x \equiv p^1/P^1$. This was expected, as all non-planar graphs are suppressed both in the weak coupling limit and in the 't Hooft model. In the weak coupling limit 't Hooft's model is equivalent to[3] light-front QED$_2$.

3   Discussion

Hadrons are the true asymptotic states of quantum chromodynamics. Hence an understanding of scattering processes requires a desciption of hadrons in arbitrary Lorentz frames. One possibility is to use wave functions defined at equal light-front time ($x^+ = 0$) [13], which have simple boost properties in the light-front direction. Light-front wave functions are moreover closely related to cross sections of hard scattering processes such as deep inelastic scattering. In the bound state rest frame the choice of a light-front direction obviously implies a loss of explicit rotational invariance.

Boost properties of bound states wave functions defined at equal x^0, such as Lorentz contraction, are usually discussed in general terms. A more explicit field theoretic description seems desirable, and could help in understanding the connection between the

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[3]Equation (45) does not contain the terms $\propto \alpha\left[1/x + 1/(1-x)\right]$ present in 't Hooft's calculation: these are a next to leading correction in the weak coupling limit.

quark and parton model pictures of hadrons. The quark model wave functions defined in the rest frame should turn into the parton model light-front wave functions when boosted to infinite momentum. As a first step in this direction we here considered the boost properties of a non-relativistic hydrogen atom in 1+1 dimensions.

Acknowledgements

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